# Regulæ
## simpliciter differentiandi generales,

quas

Consent. Ampliss. Ord. Philosoph.,

Præside


CAROLO JOH. D:s. HILL,

Math. Prof. Reg. & Ord.


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CAROLUS FREDERICUS NAUMANN,

Scanus.




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Quem in finem R. 6) æquivalentem $d(\varphi\psi)x = d\varphi(\psi x) = \varphi_1\psi x . d\psi x$ in usum vocamus, quæ quidem vice versa $d\psi x$ (ideoque $\psi_1 x$ seu $\partial\psi x$) per $d\varphi z$ (*seu* $\varphi_1 z$) & $d\varphi\psi x$ exprimit. Si igitur hæc differentialia aliquo modo invenire licet, etiam illud reperies. Speciatim vero si functiones $\varphi$ & $\psi$ tali vinculo sibi junctas ponimus, ut sit $\varphi\psi x = x$, quo casu altera alterius *inversa* dicitur, $\left\{\begin{array}{l|l} \text{\&}\ \psi x = z & = \bar{\varphi} x \\ \text{dat}\ \varphi z = x & = \bar{\psi} z \end{array}\right\}$, erit sane $d\varphi\psi x = dx$, ideoque $dx = \varphi_1\psi x . d\psi x$, seu $1 = \varphi_1\psi x . \psi_1 x$, quæ formula hanc *regulam* subministrat:

*Si derivata functionis cujusdam* $\varphi$ *datur, erit derivata ipsius inversæ* $\left(\psi_1 x = \dfrac{1}{\varphi_1\psi x}\right)$ *reciproca illius ad radicem* ($\psi x$) *relatæ.*

Sic differentiale arcuum ex ipsis functionum Trigonometricarum petas &c.

Sic, quia jam invenimus $d(a^x) = a^x La . dx$, atque hujus exponentialis $a^x$ functio inversa est Logarithmus basi $a$ gaudens, quem ideo per $\underset{a}{L}$ indicamus, ut sit $\underset{a}{L}z = x$, si $a^x = z$ & viceversa; ideoque $a^{Lz} = a^x = z$ & $\underset{a}{L}(a^x) = \underset{a}{L}z = x$, erit sane $d\underset{a}{L}z = \dfrac{dz}{zLa}$.

Si enim $\psi x = z$ ponimus, erit $\varphi z = \varphi\psi x = x$, seu $\varphi z$ *radix* ($x$) æquationis $\psi x = z$, atque $\psi x$ radix $z$ æquationis $\varphi z = x$. Similiter vero jam est $(x =)\underset{a}{L}z$ radix æquationis $a^x = z$, sicuti $(z =)a^x$ ipsius $\underset{a}{L}z = x$ est radix, prætereaque Logarithmus et exponentialis ita sibi opponuntur, ut sit $\underset{a}{L}(a^x) = x$; scripto igitur $\varphi x$ pro $a^x$ & $\psi z$ pro $\underset{a}{L}z$, erit, si $z = \varphi x = a^x$ atque $x = \underset{a}{L}z = \psi z$, $\varphi\psi z = \varphi x = a^x = z$, ideoque hæ functiones sibi adversantur, quare $\psi_1 z = \dfrac{1}{\varphi_1\psi z}$, seu cum $\varphi_1 z = a^z La$, erit $\partial\underset{a}{L}z = \partial\psi z = \psi_1 z = \dfrac{1}{a^{\psi z}La} = \dfrac{1}{a^x La}$

seu $\partial L_a^z = \frac{1}{z La}$ & $dL_a z = \frac{dz}{z.La}$ h. e. *derivata Logarithmi numero reciprocatur.* Speciatim igitur si jam per $a$ basin naturalem $\mathfrak{z}$ intelligimus, $L_a z$ in $Lz$ seu naturalem logarithmum degenerat, ideoque erit $La=1$, atque $dLz=\frac{dz}{z}$. Hinc igitur commodam differentiandi regulam adepti sumus $dz=z.dLz$, seu *differentiale quantitatis cujusvis obtineri, hanc ipsam ducendo in differentiale sui Logarithmi naturalis* (quem semper factorem superfluum $La$ evitaturi adhibemus). Sic Ex. gr. erit

1) $d(a^x)=(a^x).dL(a^x)=a^x.d(xLa)=a^x.La.dx$

2) $d(x^c)=x^c dLx^c=x^c d(cLx)=x^c.c.\frac{dx}{x}=cx^{c-1}dx$

3) $d(xyz..)=xyz..dL(xyz..)=xyz..d(Lx+Ly+Lz+..)=$
$=xyz..\left(\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}+..\right)$

4) $d\left(\frac{x}{y}\right)=\frac{x}{y}.\left(\frac{dx}{x}-\frac{dy}{y}\right)=\frac{ydx-xdy}{y^2}$ &c. . .

Similiterque functionem $\varphi$ aliter eligendo, alias ex formula $dz=\frac{d\varphi z}{\varphi_1 z}$ elicias *regulas;* hæc enim formula latissime patet.

Fieri tamen potest, ut functio proposita nullomodo in alias, cujus differentialia jam innotuerunt, resolvi possit, & ita differentiandi regula destituaris. Cui quidem rei medela quærenda est.

Quem in finem videamus, quibus modis differentiationem aggredi liceat, quibusve adjumentis hanc rem perficiamus. Quæ hæc præcipue sunt:

1) *Evolutio functionis*, quæ quidem in Algebraicis ope Theorematis Newtoni binomialis atque reversionis, cui hujus Paralellogrammum Analyticum magno semper futurum est adminiculo, facile perficitur, ideoque functiones Algebraicæ omnes ita differentiantur: utrumque enim, ut ab initio vidimus, revera unum idemque problema constituunt, seu ad maximum differentiatio simplex ab evolutione, ut pars a toto, differt.

2) Resolvendo in formas simplicissimas, cui rei formulæ & regulæ generales modo traditæ maxime inserviunt.

3) Differentias ipsius functionis successivas accuratius examinando & computando, quæ quidem res ipsi nomini *differentiali* originem tribuisse videtur.

Scilicet accidere potest, ut functio derivata inquirenda vix ex functionibus jam diu notis sit composita, vel saltem ipsius compositio adhuc nimis lateat, præsertim si originaria non nisi implicite vel ad maximum ope tabulæ constructæ innotescat. Tum quidem derivatam simili adminiculo attingas, eo scilicet ut pro valore radicis (vel argumenti variabilis) quocunque dato valorem functionis huic correspondentem computare scias.

Hæc vero ita efficias. Quia sc. $f(x+r)=fx+rf_1x+r^2f_2x+\ldots$, erit $f_1x=\frac{f\,\overline{x+r}\,fx}{r}-rf_2x-r^2f_3x$ seu $=\frac{\Delta fx}{r}-r.X$, si $\Delta fx=f\,\overline{x+r}-fx$, (quod *differentiam* functionis $fx$ vocant) & $X=f_2x+rf_3x+r^2f_4x+\ldots$, ideoque $X=f_2x$ & quidem finitum, si $x$ indeterminatum manet, & $r$ in nihilum redigitur, quo & $r.X$ in nihilum abit, ut $f_1x$ sit id, quod $\frac{\Delta fx}{r}$ tum evadit. Jam $r$ evanescente, $\Delta fx$ in $dfx$ mutari censent *methodi differentialis* auctores, atque $r$ in $dx$. Revera tamen $r=o$ efficiendum foret, quo

$\frac{\Delta fx}{r}$ in formam inditerminatam $\frac{0}{0}$ migrat, quæ quidem haud tractabili fiet, nisi antea $f\overline{x+r}$ ita evolveris, ut revera $fx$ detrahi possit, quo facto postmodum divisio per $r$ (adhuc haud evanescens) facile instituetur. Revera enim hoc exigitur, ut evolutio & divisio antea peragatur, quo facto, quotus $f_1x+rX$ aperte in $f_1x$ transit, si $r$ postmodum evanescit. Ita vero ad methodum 1:m delabimur. Sin vero hanc evolutionem ignoveris, tamen valorem ipsius $f_1x$ ex formula $\frac{\Delta fx}{r}$ (cum correctione $+rX$ perpauca) approximate saltem computabis, si $r$ minimum feceris, valoresque functionis $fx$ cognoveris. Quoniam enim $X$ tum finitum fit (& quidem ad $f_2x$ tantomagis accedit, quanto minor $x$ evadit), correctionem vel errorem $-rX$ ad libitum minues, si $r$ satis minueris, & igitur $\frac{\Delta fx}{r}$ ad $f_1x$ quantumvis placet approquinques. Sic si ignoveris valorem $\underset{c}{dLx}$, attamen tabula logarithmorum tibi ad manus est, *rationem differentialem* $\frac{dLx}{dx}$ seu $\delta Lx$ vel $L_1x$ pro valore quocunque ipsius $x$ quam proxime calculabis; omnesque ita hujusmodi valores ordine disponendo tabulam functionis $L_1x$ construeres, si aliunde hujus functionis notionem desperares. Sic posito $x=10$, & primum $r=1$, erit secundum Briggium $\frac{\Delta L10}{r}=L11-L10=0{,}04139$; sin vero $r=0{,}1$ facis, erit hæc ratio $=\frac{L10{,}1-L10}{0{,}1}=0{,}04321$, & rursum $=$ $=\frac{L10{,}01-L10}{0{,}01}=0{,}04341$ pro $r=0{,}01$ atque $=$ $0{,}0434$ pro $r=0{,}001$. Hi vero valores magis magisque ad certum numerum, cujus primæ notæ apertæ $0{,}0434$ sunt, accedunt, quem *ultimam* dixerunt *rationem*,

quæ scilicet tum locum haberet, quum $r$ evanuerit, quæque ideo nihilo ab $L_1 10$ seu valore $\partial Lx$ vel $\frac{dLx}{dx}$ pro $x=10$ differret, vel etiam ipsi $\frac{\triangle Lx}{r}$ pro $x=10$ & $r=o$ conveniret.

Eidem limiti celerius appropinquabis, si jam ab initio observaveris, esse tum $f\overline{x+r}=fx+r\partial fx+r^2\underset{c}{\partial^2}fx+r^3\underset{c}{\partial^3}fx+\ldots$

tum $f\overline{x-r}=fx-r\partial fx+r^2\underset{c}{\partial^2}fx-r^3\underset{c}{\partial^3}fx+\ldots$

ideoque $f\overline{x+r}-f\overline{x-r}=2r(\partial fx+r^2\underset{c}{\partial^3}fx+\ldots)$, quare mox erit

fere $\partial fx$ seu $f_1x=\frac{f\overline{x+r}-f\overline{x-r}}{2r}$. Sic in eodem exemplo facto $r=0{,}1$,

erit $L_1 10=\frac{L10{,}1-L9{,}9}{0{,}2}=0{,}043431$.

Attamen huic quoque formæ inest error $=-r^2\underset{c}{\partial^3}fx-r^4\underset{c}{\partial^5}fx+\ldots$, cujus valor pro $r$ minimo est fere $-r^2f_3x$ atque similiter computari potest. Quia enim $f\overline{x+r}-f\overline{x-r}=2rf_1x+2r^3f_3x+$, erit scribendo $\frac{r}{2}$ & $\frac{3}{2}r$ pro $r$, $f\overline{x+\frac{r}{2}}-f\overline{x-\frac{r}{2}}=rf_1x+\frac{r^3}{4}f_3x+\ldots$ atque $f\overline{x+\frac{3}{2}r}-f\overline{x-\frac{3}{2}r}=3rf_1x+\frac{27r^3}{4}f_3x+\ldots$, & igitur $f\overline{x+\frac{3}{2}r}-f\overline{x-\frac{3}{2}r}-3(f\overline{x+\frac{r}{2}}-f\overline{x-\frac{r}{2}})=6r^3f_3x+\ldots=r^3\partial^3fx+\ldots$, cujus formulæ ope in hoc exemplo invenies istum errorem fere $=-0{,}0000014$, ideoque $L_1 10=0{,}043430$, qui valor haud integra unitate ultimo loco a vero valore differt.

Sed hoc modo, (qui in calculo differentiarum ampliatur *) tantum-

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*) Quia $f\overline{x+r}=fx+r\partial fx+\frac{r^2\partial^2}{1.2}fx+\ldots$ & $\mathfrak{Z}^\delta=1+\delta+\frac{\delta^2}{1.2}+\frac{\delta^3}{1.2.3}+\ldots$, hæc formula, si $r\partial$ pro $\delta$ scribis, contracte & symbolice ita exhibetur $f\overline{x+r}=\mathfrak{Z}^{r\partial}fx$,

modo derivatæ valores, haud vero hujus functionis formam & quasi animam facile comprehendes. Id tamen commodi hic habet modus, ut functionem derivatam calculabilem reddat, quoties originaria talis est.

4) Ex ipsa indole vel origine functionis optime regulæ differentiandi eliciuntur. Sic si fuerit $\varphi$ functio ejus indolis, ut pro valoribus ipsorum $x$ & $y$ quibusvis sit $\varphi(xy)=\varphi x+\varphi y$, erit quidem ipsum $x$ variando & ex theoremate Tayloreo evolvendo:

$\varphi(y(x+r))=\varphi(x+r)+\varphi y$, &

$\varphi\overset{\frown}{y.x}+ry.\varphi_1\overset{\frown}{yx}+\ldots=\varphi x+r\varphi_1 x+\ldots+\varphi y$, ideoque $y\varphi_1\overset{\frown}{y.x}=\varphi_1 x$ (quod & statim derivando secundum $x$ obtinetur); similiterque variando $y$ erit $x.\varphi_1\overset{\frown}{yx}=\varphi_1 y$. Hinc igitur eliso $\varphi_1\overset{\frown}{y.x}$ communi, erit $\varphi_1 x:\varphi_1 y=y:x$ seu

$x\varphi_1 x=y\varphi_1 y$. Hoc vero convenire nequit, nisi utraque pars constans fuerit; si enim revera, ut videtur, functio ipsius $x$ functionem ipsius $y$ æquaret, etiam $y$ ipsius $x$ foret functio, ideoque sui valores ex his ipsius $x$ determinarentur, cum tamen posuerimus, ipsis $x$ & $y$ valores quosvis tribui posse. Sit igitur necessario $x.\varphi_1 x=a$, exsistente $a$ constante, (quem invenias $=c\varphi_1 c$ faciendo $y$, ut licet, $=c$); unde $\varphi_1 x=\frac{a}{x}$. Differentiale igitur hujusmodi functionis necessario erit $=d\varphi x=\frac{adx}{x}$. Jam vero vidimus, Logarithmum talem esse functionem, quare $dLx=\frac{adx}{x}$ erit. Quin imo retro

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(quod tamen sensu caret, nisi rursum simili sub forma evolvitur), ideoque erit $\Delta fx=\mathfrak{J}^{r\partial}fx-fx$ seu quasi$=(\mathfrak{J}^{r\partial}-1)fx$, & separatim $\Delta=\mathfrak{J}^{r\partial}-1$, & $1+\Delta=\mathfrak{J}^{r\partial}$, atque $L(1+\Delta)=r\partial=\Delta-\frac{\Delta^2}{2}+\frac{1}{3}\Delta^3-\ldots$, & tandem ($fx$ restituta, uti decet), $r\partial fx=$ $=\Delta fx-\frac{\Delta^2 fx}{2}+\frac{\Delta^3 fx}{3}-\ldots$ unde derivatam per differentias successivas computabis.

argues, hanc functionem $\varphi$ nihil nisi Logarithmum significare, seu maxime inde constante differre. Hoc enim *Theorema* facile demonstratur:

Si fuerit $d\varphi x = dfx$, erit necessario $\varphi x = fx +$ Const.

Universim vero si fuerit $\varphi z = \varphi x + \varphi y$, atque $z$ functio data (algebraica) ipsorum $x$ & $y$, facile $\partial\varphi x$ invenies. Primum enim $x$ constante habito, atque utrimque secundum $y$ variato, erit $\underset{y}{\partial} z . \varphi_1 z = \varphi_1 y$, tumque variando secundum $x$ erit $\underset{x}{\partial} z . \varphi_1 z = \varphi_1 x$, ideoque $\varphi_1 y : \varphi_1 x = \underset{y}{\partial} z : \underset{x}{\partial} z$, ubi necessario pars dextra in factores $X : Y$ resolvatur, quorum alter ipsius $x$ alterque ipsius $y$ sit functio, ut jam sit $\varphi_1 y : \varphi_1 x = X : Y$, ideoque $X\varphi_1 x = Y . \varphi_1 y =$ Const. $a$, unde $\varphi_1 x = \frac{a}{X}$. Sic si $z = xy$, fit $\underset{x}{\partial} z = y$ & $\underset{y}{\partial} z = x$, unde $\varphi_1 z = \frac{a}{x}$, ut modo invenimus.

Sic si $z = \frac{x+y}{1-xy}$, fit $\underset{x}{\partial} z = \frac{(1-xy)+(x+y)y}{(1-xy)^2} = \frac{1+y^2}{\overline{1-xy}^2}$ & $\underset{y}{\partial} z = \frac{1+x^2}{\overline{1-xy}^2}$, quare $\varphi_1 x : \varphi_1 y = (1+y^2) : (1+x^2)$, & $\varphi_1 x = \frac{a}{1+x^2}$.

Est vero $\varphi \frac{x+y}{1-xy} = \varphi x + \varphi y$, si $\varphi x$ arcum, cujus tangens est $x$, significat, id quod per $\perp x$ indicamus, ideoque $d \perp x = \frac{adx}{1+x^2}$, ubi const. $a$ ex mensura arcus accuratius definiatur (nempe $a = 1$, radio $= 1$).

Sed hæc eadem res etiam hoc perficitur modo. Quia $z$ functio est ipsorum $x$ & $y$, quam per $(xy)$ indicare licet, & $x$ atque $y$ arbitrio relinquuntur, faciamus $z = x + rdx$, resolvamusque æquationem $(z =)\ x + rdx = (xy)$ secundum $y$, ut habeatur $y = (\overline{x}z) = (\overline{x}(x + r\partial x))$, quem valorem se-

cundum $r$ evolvamus, ut jam sit $y=X+rX_1dx+r^2X_2\widehat{dx^2}+\ldots$, (id quod certe tum ad minimum succedit, cum $z$ vel $xy$ functio est algebraica). Quia igitur $\varphi z=\varphi x+\varphi y$, erit jam $\varphi(x+rdx)-\varphi x=\varphi(X+rX_1+\ldots)$ seu $\varphi_1 x.rdx+\varphi_2 x.\overline{rdx}^2+ = \varphi X+rX_1dx.\varphi_1 X+\ldots$

Hinc igitur patet, partim $X$ talem necessario esse, ut sit $\varphi X=o$, ideoque $X$ constans, partimque $\varphi_1 x=X_1.\varphi_1 X$, ideoque $\varphi_1 x=X_1.C$, existente $C=\varphi_1 X=$constante.

Sic si $(xy)=x.y$ fuerit, faciendum erit $x+rdx=x.y$, unde $y=\frac{x+rdx}{x}=1+r.\frac{dx}{x}$, ideoque $X=1$, $X_1=\frac{1}{x}$, & igitur $\varphi_1 x=\frac{C}{x}=\partial\varphi x$, seu $\partial Lx=\frac{C}{x}$, ut antea, (quia tum $\varphi$ in logarithmum degenerat).

Sic si $(xy)=b^1x+b_1y+cxy=z$ foret, haberetur $y=(\bar{x}z)=\frac{z-b^1x}{b_1+cx}=$ $=\frac{x+rdx-b^1x}{b_1+cx}=\frac{x(1-b^1)}{b_1+cx}+\frac{rdx}{b_1+cx}$ & $=X+rX_1dx$, ideoque $X$ haud constans, nisi $b^1=1$ vel $b_1=o$, atque $X_1=\frac{1}{b_1+cx}$. Quia vero $\varphi z=\varphi x+\varphi y=$ $=\varphi y+\varphi x$, erit necessario $z$ symmetrica ad $x$ & $y$, ideoque $b^1=b_1=b$, & igitur $b$ vel $=o$, vel $=1$. Illo casu $(xy)=cxy$, & $X_1=\frac{1}{cx}$ evadit, ideoque $\varphi_1 x=\frac{C}{cx}$. Hoc vero, seu cum $b=1$, fit $z=x+y+cxy$, $\varphi(x+y+cxy)$ $=\varphi x+\varphi y$ atque $\varphi_1 x=\frac{Cc}{1+cx}$. Quin imo hinc functio $\varphi$ hac ipsa proprietate gaudens invenitur. Quia enim $\partial L(1+cx)=\frac{\partial(1+cx)}{1+cx}=\frac{c}{1+cx}$, erit $\partial\varphi x=C.\partial L(1+cx)$, quare rederivando vel *integrando*, ut dicunt, obtinetur $\varphi x=CL\overline{1+cx}+A$.